

SOLVABLE LEIBNIZ ALGEBRAS WITH NATURALLY GRADED NON-LIE p -FILIFORM NILRADICALS

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ABSTRACT. In this paper solvable Leibniz algebras with naturally graded non-Lie p -filiform ($n - p \geq 4$) nilradical and with one-dimensional complemented space of nilradical are described. Moreover, solvable Leibniz algebras with abelian nilradical and extremal (minimal, maximal) dimensions of complemented space nilradical are studied. The rigidity of solvable Leibniz algebras with abelian nilradical and maximal dimension of its complemented space is proved.

1. INTRODUCTION

Leibniz algebras are generalizations of Lie algebras and they have been introduced by Loday in [17] as a non-antisymmetric version of Lie algebras. These algebras preserve a unique property of Lie algebras - the right multiplication operators are derivations. Since the 1993 when Loday's work was published, many researchers have been attracted to Leibniz algebras, with remarkable activity during the last decades.

From the classical theory of Lie algebras it is well known that the study of finite-dimensional Lie algebras was reduced to the nilpotent ones [19]. In the Leibniz algebra case we have an analogue of Levi's theorem [6]. Namely, the decomposition of a Leibniz algebra into a semidirect sum of its solvable radical and a semisimple Lie algebra is obtained. The semisimple part can be described from simple Lie ideals (see [6]) and therefore, the main problem is to study the solvable radical. Based on the work of [20], a new approach for the investigation of solvable Lie algebras by using their nilradicals is developed in the works [1–3, 21, 23–25] and others. This approach is based on the nil-independent derivations of nilradical. In fact, for a given solvable Leibniz algebra with a fixed nilradical the complemented space to nilradical has a basis with the condition that the restriction of operators of right multiplication on a basis element is nil-independent derivation of nilradical [12].

Since the description of finite-dimensional solvable Lie algebras is a boundless problem, lately geometric approaches are developing. Relevant tools of geometric approaches are Zariski topology and natural action of linear reductive group on varieties of algebras in a such way that orbits under the action consists of isomorphic algebras. It is a well-known result of algebraic geometry that any algebraic variety (evidently, algebras defined via identities form an algebraic variety) is a union of a finite number of irreducible components. The most important algebras are those whose orbits under the action are open sets in sense of Zariski topology. The algebras of a variety with open orbits are important since the closures of orbits of such algebras form irreducible components of the variety. At the same time there exist an irreducible component which is not closure of orbit of any algebra. This fact does not detract the importance of algebras with open orbits. This is a motivation of many works focused to discovering of algebras with open orbits and to description of sufficient properties of such algebras [7, 15, 16].

In this paper under the condition to a operator of right multiplication we classify solvable Leibniz algebras with abelian nilradical and with one-dimensional complemented space of nilradical. The classification of solvable Leibniz algebras with abelian nilradical and maximal dimension of complemented

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space is obtained, as well. The rigidity of such algebras is proved. Moreover, we present a description of solvable algebras with naturally graded non-Lie p -filiform nilradicals ($n - p \geq 4$) under some condition to operator of right multiplication to an element of one-dimensional complemented space to nilradical. The classifications problems of obtained algebras of the descriptions are studied.

Throughout the paper we consider finite-dimensional vector spaces and algebras over the field \mathbb{C} . Moreover, in the multiplication table of an algebra omitted products are assumed to be zero and if it is not noticed we shall consider non-nilpotent solvable algebras.

2. PRELIMINARIES

In this section we give necessary definitions and preliminary results.

Definition 2.1 ([17]). A vector space with bilinear bracket $(L, [-, -])$ over a field \mathbb{F} is called a *Leibniz algebra* if for any $x, y, z \in L$ the so-called Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

holds, or equivalently, $[[x, y], z] = [[x, z], y] + [x, [y, z]]$.

Here, we adopt the right Leibniz identity; since the bracket is not skew-symmetric, there exists the version corresponding to the left Leibniz identity,

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]].$$

For examples of Leibniz algebras we refer to papers [17] and [18].

Further we will use the notation

$$\mathcal{L}(x, y, z) = [x, [y, z]] - [[x, y], z] + [[x, z], y].$$

It is obvious that Leibniz algebras are determined by the identity $\mathcal{L}(x, y, z) = 0$.

From the Leibniz identity we conclude that the elements $[x, x], [x, y] + [y, x]$ for any $x, y \in L$ lie in $\text{Ann}_r(L) = \{x \in L \mid [y, x] = 0, \text{ for all } y \in L\}$, the *right annihilator* of the Leibniz algebra L . Moreover, it is easy to see that $\text{Ann}_r(L)$ is a two-sided ideal of L .

The two-sided ideal $\text{Center}(L) = \{x \in L \mid [x, y] = 0 = [y, x], \text{ for all } y \in L\}$ is said to be the *center* of L .

Definition 2.2. A linear map $d: L \rightarrow L$ of a Leibniz algebra $(L, [-, -])$ is said to be a *derivation* if for all $x, y \in L$ the following condition holds:

$$d([x, y]) = [d(x), y] + [x, d(y)]. \quad (2.1)$$

Note that the right multiplication operator $\mathcal{R}_x: L \rightarrow L, \mathcal{R}_x(y) = [y, x], y \in L$, is a derivation (for a left Leibniz algebra L , the left multiplication operator $\mathcal{L}_x: L \rightarrow L, \mathcal{L}_x(y) = [x, y], y \in L$, is also derivations).

Definition 2.3. For a given Leibniz algebra $(L, [-, -])$ the sequences of two-sided ideals defined recursively as follows:

$$L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \geq 1, \quad L^{[1]} = L, \quad L^{[s+1]} = [L^{[s]}, L^{[s]}], \quad s \geq 1,$$

are said to be the *lower central* and the *derived series* of L , respectively.

Definition 2.4. A Leibniz algebra L is said to be *nilpotent* (respectively, *solvable*), if there exists $n \in \mathbb{N}$ ($m \in \mathbb{N}$) such that $L^n = 0$ (respectively, $L^{[m]} = 0$). The minimal number n (respectively, m) with such property is said to be the *index of nilpotency* (respectively, *index of solvability*) of the algebra L .

Evidently, the index of nilpotency of an n -dimensional nilpotent algebra is not greater than $n + 1$.

Definition 2.5. The maximal nilpotent ideal of a Leibniz algebra is said to be the *nilradical* of the algebra.

Let R be a solvable Leibniz algebra with nilradical N . We denote by Q the complementary vector space of the nilradical N to the algebra R . Let us consider the restrictions to N of the right multiplication operator on an element $x \in Q$ (denoted by $\mathcal{R}_{x|_N}$). From [12] we know that for any $x \in Q$, the operator $\mathcal{R}_{x|_N}$ is a non-nilpotent derivation of N .

Let $\{x_1, \dots, x_m\}$ be a basis of Q , then for any scalars $\{\alpha_1, \dots, \alpha_m\} \in \mathbb{C} \setminus \{0\}$, the matrix $\alpha_1 \mathcal{R}_{x_1|_N} + \dots + \alpha_m \mathcal{R}_{x_m|_N}$ is non-nilpotent, which means that the elements $\{x_1, \dots, x_m\}$ are nil-independent [20].

Therefore, we have that the dimension of Q is bounded by the maximal number of nil-independent derivations of the nilradical N (see [12, Theorem 3.2]). Moreover, similar to the case of Lie algebras, for a solvable Leibniz algebra R the inequality $\dim N \geq \frac{\dim R}{2}$ holds.

Let L be a nilpotent Leibniz algebra and $x \in L \setminus L^2$. Denote by $C(x) = (n_1, n_2, \dots, n_k)$ the decreasing sequence which consists of the dimensions of the Jordan blocks of the operator R_x . On the set of such sequences we consider the lexicographic order.

Definition 2.6. The sequence $C(L) = \max_{x \in L \setminus L^2} C(x)$ is called the *characteristic sequence* of the Leibniz algebra L .

Below we define the notion p -filiform Leibniz algebra.

Definition 2.7. A Leibniz algebra L is called *p -filiform* if $C(L) = (n - p, \underbrace{1, \dots, 1}_p)$, where $p \geq 0$.

Note that above definition, when $p > 0$ agrees with the definition of p -filiform Lie algebras [8]. Since in the case of Lie algebras there is no singly-generated algebra, the notion of 0-filiform algebra for Lie algebras has no sense, while for the Leibniz algebras case in each dimension there exists up to isomorphism a unique null-filiform algebra [4].

Definition 2.8. Given an n -dimensional p -filiform Leibniz algebra L , put $L_i = L^i / L^{i+1}$, $1 \leq i \leq n - p$, and $\text{gr } L = L_1 \oplus L_2 \oplus \dots \oplus L_{n-p}$. Then $[L_i, L_j] \subseteq L_{i+j}$ and we obtain the graded algebra $\text{gr } L$. If $\text{gr } L$ and L are isomorphic, $\text{gr } L \cong L$, we say that L is *naturally graded*.

Due to voluminous of the list of naturally graded p -filiform ($1 \leq p \leq n - 1$) Lie algebras we refer the reader to the work [9].

Since we shall consider naturally graded non-Lie p -filiform nilradical we give its classification.

Theorem 2.9 ([10]). *An arbitrary n -dimensional naturally graded non-split p -filiform Leibniz algebra ($n - p \geq 4$) is isomorphic to one of the following non-isomorphic algebras:
 $p = 2k$ is even*

$$\begin{aligned} \mu_1 : & \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n - 2k - 1, \\ [e_1, f_j] = f_{k+j}, & 1 \leq j \leq k, \end{cases} \\ \mu_2 : & \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n - 2k - 1, \\ [e_1, f_1] = e_2 + f_{k+1}, \\ [e_i, f_1] = e_{i+1}, & 2 \leq i \leq n - 2k - 1, \\ [e_1, f_j] = f_{k+j}, & 2 \leq j \leq k, \end{cases} \end{aligned}$$

$p = 2k + 1$ is odd

$$\mu_3 : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n - 2k - 2, \\ [e_1, f_j] = f_{k+1+j}, & 1 \leq j \leq k, \\ [e_i, f_{k+1}] = e_{i+1}, & 1 \leq i \leq n - 2k - 2, \end{cases}$$

where $\{e_1, e_2, \dots, e_{n-p}, f_1, f_2, \dots, f_p\}$ is a basis of the algebra.

In order to simplify our further calculations for the algebra μ_3 , by taking the change of basis in the following form:

$$e'_1 = e_1, \quad e'_2 = e_1 - f_{k+1}, \quad e'_{i+1} = e_i, \quad 2 \leq i \leq n - 2k - 1, \quad f'_j = f_j, \quad f'_{k+j} = f_{k+1+j}, \quad 1 \leq j \leq k,$$

we obtain the table of multiplication of the algebra μ_3 , which we shall use throughout the paper:

$$\mu_3 : \begin{cases} [e_1, e_1] = e_3, \\ [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n - 2k - 1, \\ [e_1, f_j] = f_{k+j}, & 1 \leq j \leq k, \\ [e_2, f_j] = f_{k+j}, & 1 \leq j \leq k. \end{cases}$$

For acquaintance with the definition of cohomology group of Leibniz algebras and its applications to the description of the variety of Leibniz algebras (similar to Lie algebras case) we refer the reader to

the papers [5, 13, 14, 17, 18, 22]. Here we just recall that the second cohomology group of a Leibniz algebra L with coefficients in a corepresentation M is the quotient space

$$HL^2(L, M) = ZL^2(L, M)/BL^2(L, M),$$

where the 2-cocycles $\varphi \in ZL^2(L, M)$ and the 2-coboundaries $f \in BL^2(L, M)$ are defined as follows

$$(d^2\varphi)(a, b, c) = [a, \varphi(b, c)] - [\varphi(a, b), c] + [\varphi(a, c), b] + \varphi(a, [b, c]) - \varphi([a, b], c) + \varphi([a, c], b) = 0 \quad (2.2)$$

and

$$f(a, b) = [d(a), b] + [a, d(b)] - d([a, b]) \text{ for some linear map } d. \quad (2.3)$$

The linear reductive group $\mathrm{GL}_n(\mathbb{F})$ acts on \mathbf{Leib}_n , the variety of n -dimensional Leibniz algebra structures, via change of basis, i.e.,

$$(g * \lambda)(x, y) = g\left(\lambda(g^{-1}(x), g^{-1}(y))\right), \quad g \in \mathrm{GL}_n(\mathbb{F}), \lambda \in \mathbf{Leib}_n.$$

The orbits $\mathrm{Orb}(-)$ under this action are the isomorphism classes of algebras. Recall, Leibniz algebras with open orbits are called *rigid*. Note that solvable Leibniz algebras of the same dimension also form an invariant subvariety of the variety of Leibniz algebras under the mentioned action.

Remark 2.10. Due to results of the paper [4] we have a sufficient condition for a Leibniz algebra being rigid algebra. Namely, if the second cohomology of a Leibniz algebra with coefficients in itself is trivial, then it is a rigid algebra.

3. SOLVABLE LEIBNIZ ALGEBRAS WITH ABELIAN NILRADICAL AND EXTREMAL DIMENSIONS OF COMPLEMENTED SPACE Q

We denote by $\mathcal{A}(k)$ the k -dimensional abelian algebras. For solvable Leibniz algebras with nilradical $\mathcal{A}(k)$ and dimension of complemented space of nilradical to an algebra is equal to s , we shall use the notation $R(\mathcal{A}(k), s)$.

This section is devoted to the classification of solvable Leibniz algebras with nilradical $\mathcal{A}(k)$ under the condition that the complemented space to the nilradical have extremal dimensions. The extremal dimensions of the complemented space means the minimal and maximal possible dimensions of the space. Evidently, the candidate for minimal dimension of complemented space is equal one.

Firstly, we consider the case of solvable algebras with one-dimensional complemented space of $\mathcal{A}(k)$.

Let $\{f_1, f_2, \dots, f_k, x\}$ be a basis of the algebra $R(\mathcal{A}(k), 1)$.

Evidently, the space of derivations of the algebra $\mathcal{A}(k)$ coincided with the space of $k \times k$ matrices.

Let us assume that the operator $\mathcal{R}_{x|_{\mathcal{A}(k)}}$ has Jordan block form, that is, $\mathcal{R}_{x|_{\mathcal{A}(k)}} = J_\lambda$.

Theorem 3.1. *An arbitrary algebra of the family $R(\mathcal{A}(k), 1)$ is isomorphic to one of the following non-isomorphic algebras:*

$$R_1 : \begin{cases} [f_i, x] = f_i + f_{i+1}, & 1 \leq i \leq k-1, \\ [f_k, x] = f_k, \end{cases} \quad R_2 : \begin{cases} [f_i, x] = f_i + f_{i+1}, & 1 \leq i \leq k-1, \\ [f_k, x] = f_k, \\ [x, f_i] = -f_i - f_{i+1}, & 1 \leq i \leq k-1, \\ [x, f_k] = -f_k. \end{cases}$$

Proof. Since $\lambda = 0$ implies the nilpotency of the algebra R we get $\lambda \neq 0$. Moreover, by scaling the basis elements

$$x' = \lambda^{-1}x, \quad f'_i = \lambda^{1-i}f_i, \quad 1 \leq i \leq k,$$

we can suppose $\lambda = 1$.

Therefore, the multiplication table of the algebra R has the following form:

$$\begin{cases} [f_i, x] = f_i + f_{i+1}, & 1 \leq i \leq k-1, \\ [f_k, x] = f_k, \\ [x, f_i] = \sum_{j=1}^k \alpha_{i,j} f_j, & 1 \leq i \leq k-1, \\ [x, x] = \sum_{i=1}^k \theta_i f_i. \end{cases}$$

From the equalities $\mathcal{L}(x, f_i, x) = \mathcal{L}(x, x, f_1) = 0$ with $1 \leq i \leq k$, derive the restrictions:

$$\begin{cases} \alpha_{i,j} = 0, & 1 \leq j < i \leq k, \\ \alpha_{i,j} = \alpha_{1,j-i+1}, & 1 \leq i \leq j \leq k, \\ \alpha_{1,1}^2 = -\alpha_{1,1}, \\ \sum_{i=1}^j \alpha_{1,i} \alpha_{1,j-i+1} = -\alpha_{1,j-1} - \alpha_{1,j}, & 2 \leq j \leq k. \end{cases} \quad (3.1)$$

Consider the possible cases.

Case 1. Let $\alpha_{1,1} = 0$. Then from the fourth equation of restrictions (3.1) we get $\alpha_{1,i} = 0$, $2 \leq i \leq k$. By taking the change in the following form:

$$x' = x + \sum_{i=1}^k \sum_{j=1}^i (-1)^{i+j-1} \theta_j f_i$$

we obtain the algebra R_1 .

Case 2. Let $\alpha_{1,1} = -1$. Then from restrictions (3.1) we obtain $\alpha_{1,2} = -1$ and $\alpha_{1,i} = 0$, $3 \leq i \leq k$. The equality $\mathcal{L}(x, x, x) = 0$ implies $\theta_i = 0$, $1 \leq i \leq k$. Thus, we obtain the algebra R_2 \square

Now we shall classify the opposite case to one-dimensional complemented space Q , that is, we consider the maximal dimension of Q .

Theorem 3.2. *The maximal possible dimension of algebras of the family $R(\mathcal{A}(k), s)$ is equal to $2k$, that is, $s = k$. Moreover, an arbitrary algebra of the family $R(\mathcal{A}(k), k)$ is decomposed into a direct sum of copies of two-dimensional non-trivial solvable Leibniz algebras.*

Proof. Let $\{x_1, x_2, \dots, x_s\}$ be a basis of the complemented space of $\mathcal{A}(k)$ to the algebra $R(\mathcal{A}(k), s)$.

Then from Leibniz identity

$$0 = [f_i, [x_j, x_t]] = [[f_i, x_j], x_t] - [[f_i, x_t], x_j]$$

we conclude that the operators $\mathcal{R}_{x_i|_{\mathcal{A}(k)}}$, $1 \leq i \leq s$, commute pairwise, that is, $\mathcal{R}_{x_j|_{\mathcal{A}(k)}} \circ \mathcal{R}_{x_t|_{\mathcal{A}(k)}} = \mathcal{R}_{x_t|_{\mathcal{A}(k)}} \circ \mathcal{R}_{x_j|_{\mathcal{A}(k)}}$ for any $1 \leq j, t \leq s$. This implies that all operators $\mathcal{R}_{x_i|_{\mathcal{A}(k)}}$, $1 \leq i \leq s$ could be simultaneously transformed to their Jordan forms by a basis transformation.

Let $\{\lambda_1^{(i)}, \lambda_2^{(i)}, \dots, \lambda_k^{(i)}\}$ be the eigenvalues of the operators corresponding to $\mathcal{R}_{x_i|_{\mathcal{A}(k)}}$, $1 \leq i \leq s$.

Consider the vectors $\alpha_i = (\lambda_1^{(i)}, \lambda_2^{(i)}, \dots, \lambda_k^{(i)})$, $1 \leq i \leq s$, of the k -dimensional vector space \mathbb{C}^k . Since $\text{Der}(\mathcal{A}(k)) \cong M_{k,k}$ and nil-independent derivations are $\mathcal{R}_{x_i|_{\mathcal{A}(k)}}$, $1 \leq i \leq s$. We deduce that the maximal number of nil-independent among vectors α_i , $1 \leq i \leq s$ is equal to k . This means that the maximal dimension of the complemented space is equal to k , i.e. $s = k$.

Without loss of generality we can assume that $\alpha_i = (0, \dots, 0, \underbrace{\lambda_i}_{i\text{-th place}}, 0, \dots, 0)$, $\lambda_i \neq 0$, $1 \leq i \leq k$,

correspond to $\mathcal{R}_{x_i|_{\mathcal{A}(k)}}$, $1 \leq i \leq k$.

Thus, we obtain the products in the algebras $R(\mathcal{A}(k), k)$:

$$\begin{cases} [f_i, x_j] = a_{i,j} f_{i+1}, & 1 \leq i \leq k-1, 1 \leq j \leq k, i \neq j, \\ [f_i, x_i] = \lambda_i f_i + a_{i,i} f_{i+1}, & 1 \leq i \leq k-1, \\ [f_k, x_k] = \lambda_k f_k, \\ [f_k, x_j] = 0, & 1 \leq j \leq k-1. \end{cases}$$

By scaling basis elements $x'_i = \frac{1}{\lambda_i} x_i$, $1 \leq i \leq k$, we obtain $\lambda_i = 1$.

The equalities

$$\mathcal{L}(f_i, x_i, x_j) = \mathcal{L}(f_i, x_i, x_{i+1}) = 0$$

derive $a_{i,j} = 0$, $1 \leq i \leq k-1$, $1 \leq j \leq k$.

Let us introduce the notations

$$[x_i, f_j] = \sum_{t=1}^k \alpha_{i,j}^t f_t, \quad [x_i, x_j] = \sum_{t=1}^k \gamma_{i,j}^t f_t, \quad 1 \leq i, j \leq k.$$

From $\mathcal{L}(x_i, f_j, x_j) = 0$ we have $[x_i, f_j] = \alpha_{i,j} f_j$.

Moreover, the equalities $\mathcal{L}(x_i, x_j, f_j) = \mathcal{L}(x_i, x_j, f_i) = 0$ imply the restrictions:

$$\begin{cases} \alpha_{i,j} = 0, & 1 \leq i, j \leq k, \quad i \neq j, \\ \alpha_{i,i}^2 = -\alpha_{i,i}, & 1 \leq i \leq k. \end{cases}$$

Consider the chain of equalities

$$[x_i, [x_j, x_i]] = [[x_i, x_j], x_i] - [[x_i, x_i], x_j] = \gamma_{i,j}^i f_i - \gamma_{i,i}^j f_j.$$

On the other hand, we have

$$[x_i, [x_j, x_i]] = \left[x_i, \sum_{t=1}^k \gamma_{j,i}^t f_t \right] = \alpha_{i,i} \gamma_{j,i}^i f_i.$$

By comparing the coefficients at the basis elements we obtain

$$\begin{cases} \alpha_{i,i} \gamma_{j,i}^i = \gamma_{i,j}^i, & 1 \leq i, j \leq k, \quad i \neq j, \\ \alpha_{i,i} \gamma_{i,i}^i = 0, & 1 \leq i \leq k, \quad i = j, \\ \gamma_{i,i}^j = 0, & 1 \leq i, j \leq k, \quad i \neq j. \end{cases} \quad (3.2)$$

We can assume that $\gamma_{i,i}^i = 0$ for $1 \leq i \leq k$. Indeed, if $\alpha_{i,i} \neq 0$ for some i , then from the above restrictions we have $\gamma_{i,i}^i = 0$. For those i such that $\alpha_{i,i} = 0$ by taking the change $x'_i = x_i - \gamma_{i,i}^i f_i$, we again obtain $\gamma_{i,i}^i = 0$. Therefore, $\gamma_{i,i}^i = 0$, $1 \leq i \leq k$, that is, $[x_i, x_i] = 0$, $1 \leq i \leq k$.

Consider the chain of equalities

$$[x_i, [x_j, x_t]] = [[x_i, x_j], x_t] - [[x_i, x_t], x_j] = \gamma_{i,j}^t f_t - \gamma_{i,t}^j f_j.$$

On the other hand, we have

$$[x_i, [x_j, x_t]] = \left[x_i, \sum_{l=1}^k \gamma_{j,t}^l f_l \right] = \alpha_{i,i} \gamma_{j,t}^i f_i.$$

From these we obtain

$$\gamma_{i,j}^t = 0, \quad 1 \leq i, j, t \leq k, \quad i \neq j, \quad i \neq t, \quad j \neq t.$$

By taking the change of basis element:

$$x'_i = x_i - \sum_{t=1, t \neq i}^k \gamma_{i,t}^t f_t, \quad 1 \leq i \leq k,$$

and by taking into account restrictions (3.2) we can conclude that $[x_i, x_j] = 0$, $1 \leq i, j \leq k$.

Thus, we have the multiplication table of the family of algebras $R(\mathcal{A}(k), k)$:

$$\begin{cases} [f_i, x_i] = f_i, & 1 \leq i \leq k, \\ [x_i, f_i] = \alpha_{i,i} f_i, & 1 \leq i \leq k, \end{cases}$$

where $\alpha_{i,i}^2 = -\alpha_{i,i}$, $1 \leq i \leq k$. □

Remark 3.3. The number of non-isomorphic algebras in the family $R(\mathcal{A}(k), k)$ is equal to $k + 1$.

It should be noted that in the work [11] the algebras $R(\mathcal{A}(2), 2)$ were already classified.

Let $l_2 : [e, x] = e$ and $r_2 : [e, x] = e, [x, e] = -e$ be two-dimensional non-Lie Leibniz and Lie algebras.

Consider the algebra $\mathcal{L}_t = l_2 \oplus l_2 \oplus \cdots \oplus l_2 \oplus r_2 \oplus r_2 \oplus \cdots \oplus r_2$, where t ($0 \leq t \leq k$) is the number of entries of l_2 in \mathcal{L}_t . Then there exists a basis $\{e_1, e_2, \dots, e_t, x_1, x_2, \dots, x_t, y_1, y_2, \dots, y_{k-t}, f_1, f_2, \dots, f_{k-t}\}$ of \mathcal{L}_t such that the multiplication table has the form:

$$[e_i, x_i] = e_i, \quad 1 \leq i \leq t, \quad [f_j, y_j] = -[y_j, f_j] = f_j, \quad 1 \leq j \leq k - t.$$

Let us present the general form of a derivation of the algebra \mathcal{L}_t .

Proposition 3.4. Any derivation d of the algebra from \mathcal{L}_t has the following form:

$$d(e_i) = a_i e_i, \quad 1 \leq i \leq t, \quad d(f_j) = b_j f_j, \quad d(y_j) = c_j f_j, \quad 1 \leq j \leq k - t.$$

Proof. The proof is carrying out by straightforward verification of derivation property (2.1). □

Now we could to easily calculate the dimensions of the spaces $\text{Der}(\mathcal{L}_t)$.

Corollary 3.5.

$$\dim \text{Der}(\mathcal{L}_t) = 2k - t, \quad \dim BL^2(\mathcal{L}_t, \mathcal{L}_t) = 4k^2 - 2k + t.$$

In order to prove the triviality of the second group of cohomology for the algebra \mathcal{L}_t with coefficients in itself we need to describe the space of 2-cocycles.

Proposition 3.6. *Any element of $\varphi \in ZL^2(\mathcal{L}_t, \mathcal{L}_t)$ has the following form:*

$$\begin{aligned} \varphi(x_i, x_j) &= \alpha_{i,j} e_j, & 1 \leq i, j \leq t, \\ \varphi(e_i, x_i) &= \sum_{m=1}^t \beta_{i,m}^1 x_m + \sum_{m=1}^t \beta_{i,m}^2 e_m + \sum_{m=1}^{k-t} \beta_{i,m}^3 y_m + \sum_{m=1}^{k-t} \beta_{i,m}^4 f_m, & 1 \leq i \leq t, \\ \varphi(e_j, x_i) &= -\beta_{i,j}^2 e_i + \nu_{j,i} e_j, & 1 \leq i, j \leq t, \ i \neq j, \\ \varphi(y_j, x_i) &= \gamma_{j,i}^1 e_i + \gamma_{j,i}^2 f_j, & 1 \leq i \leq t, \ 1 \leq j \leq k-t, \\ \varphi(y_j, y_i) &= -\varphi(y_i, y_j) = \delta_{j,i}^1 f_i + \delta_{j,i}^2 f_j, & 1 \leq i < j \leq k-t, \\ \varphi(f_i, y_i) &= \sum_{m=1}^t \xi_{i,m}^1 x_m + \sum_{m=1}^t \xi_{i,m}^2 e_m + \sum_{m=1}^{k-t} \xi_{i,m}^3 y_m + \sum_{m=1}^{k-t} \xi_{i,m}^4 f_m, & 1 \leq i \leq k-t, \\ \varphi(f_j, y_i) &= -\varphi(y_i, f_j) = -\xi_{i,j}^4 f_i + \eta_{j,i} f_j, & 1 \leq i, j \leq k-t, \ i \neq j, \\ \varphi(f_j, x_i) &= -\xi_{i,j}^2 e_i + \theta_{j,i} f_j, & 1 \leq i \leq t, \ 1 \leq j \leq k-t, \\ \varphi(e_i, y_j) &= -\beta_{i,j}^4 f_j + \tau_{j,i} e_j, & 1 \leq i \leq t, \ 1 \leq j \leq k-t, \\ \varphi(x_i, y_j) &= -\gamma_{i,j}^2 f_j, & 1 \leq i \leq t, \ 1 \leq j \leq k-t, \\ \varphi(e_i, e_j) &= -\beta_{j,i}^1 e_i, & 1 \leq i, j \leq t, \\ \varphi(f_j, e_i) &= -\beta_{i,j}^3 f_j, & 1 \leq i \leq t, \ 1 \leq j \leq k-t, \\ \varphi(y_j, e_i) &= \beta_{j,i}^4 f_j, & 1 \leq i \leq t, \ 1 \leq j \leq k-t, \\ \varphi(f_i, f_j) &= -\varphi(f_j, f_i) = -\xi_{i,j}^3 f_i + \xi_{j,i}^3 f_j, & 1 \leq j < i \leq k-t, \\ \varphi(e_i, f_j) &= -\xi_{j,i}^1 e_i + \beta_{i,j}^3 f_j, & 1 \leq i \leq t, \ 1 \leq j \leq k-t, \\ \varphi(x_i, f_j) &= -\theta_{j,i} f_j, & 1 \leq i \leq t, \ 1 \leq j \leq k-t. \end{aligned}$$

Proof. The proof is carrying out by straightforward calculations of equations (2.2) on the basis elements of the algebra \mathcal{L}_t . \square

As consequence from Proposition 3.6 we have the following corollary.

Corollary 3.7.

$$\dim ZL^2(\mathcal{L}_t, \mathcal{L}_t) = 4k^2 - 2k + t, \quad \dim HL^2(\mathcal{L}_t, \mathcal{L}_t) = 0.$$

Now we give the main result regarding the rigidity of algebras \mathcal{L}_t .

Theorem 3.8. *The algebra \mathcal{L}_t is rigid algebra for any values of t ($0 \leq t \leq k$).*

Proof. The proof of the theorem completes the argumentation of Remark 2.10 and Corollary 3.7. \square

4. SOLVABLE $n+1$ -DIMENSIONAL LEIBNIZ ALGEBRAS WITH n -DIMENSIONAL NATURALLY GRADED NON-LIE p -FILIFORM NILRADICALS.

In this section we describe solvable Leibniz algebra with naturally graded non-Lie p -filiform nilradicals under the condition $\dim Q = 1$. We focus in the non-split p -filiform non-Lie Leibniz algebras case.

4.1. Derivations of algebras $\mu_i, i = 1, 2, 3$.

In order to start the description we need to know the derivations of naturally graded non-Lie p -filiform Leibniz algebras.

Proposition 4.1. *Any derivation of the algebra μ_1 has the following matrix form:*

$$\mathbb{D} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$A = \sum_{i=1}^{n-2k} ia_1 e_{i,i} + \sum_{i=1}^{n-2k-1} \sum_{j=i+1}^{n-2k} a_{j-i+1} e_{i,j}, \quad B = \sum_{i=1}^{2k} b_i e_{1,i} + \sum_{i=1}^k b_i e_{2,k+i},$$

$$C = \sum_{i=1}^k c_i e_{i,n-2k}, \quad D = \begin{pmatrix} D_1 & D_2 \\ 0 & a_1 \mathbb{E} + D_1 \end{pmatrix},$$

$A \in M_{n-2k,n-2k}$, $B \in M_{n-2k,2k}$, $C \in M_{2k,n-2k}$, $D_1, D_2, \mathbb{E} \in M_{k,k}$ and matrix units $e_{i,j}$.

Proof. Let $\{e_1, f_1, f_2, \dots, f_k\}$ be a generator basis elements of the algebra μ_1 .

We put

$$d(e_1) = \sum_{i=1}^{n-2k} a_i e_i + \sum_{i=1}^{2k} b_i f_i, \quad d(f_i) = \sum_{j=1}^{n-2k} c_{i,j} e_j + \sum_{j=1}^{2k} d_{i,j} f_j, \quad 1 \leq i \leq k.$$

From the derivation property (2.2) we have

$$d(e_2) = d([e_1, e_1]) = [d(e_1), e_1] + [e_1, d(e_1)] = 2a_1 e_2 + \sum_{i=3}^{n-2k} a_{i-1} e_i + \sum_{i=1}^k b_i f_{k+i}.$$

Buy applying the induction and the derivation property (2.2) we derive

$$d(e_i) = i a_1 e_i + \sum_{t=i+1}^{n-2k} a_{t-i+1} e_t, \quad 3 \leq i \leq n-2k.$$

Consider

$$0 = d([f_i, e_1]) = [d(f_i), e_1] + [f_i, d(e_1)] = \sum_{j=1}^{n-2k-1} c_{i,j} e_{j+1}, \quad 1 \leq i \leq k.$$

Consequently,

$$c_{i,j} = 0, \quad 1 \leq i \leq k, \quad 1 \leq j \leq n-2k-1.$$

Similarly, from $d(f_{k+i}) = d([e_1, f_i])$, $1 \leq i \leq k$, we deduce

$$d(f_{k+i}) = a_1 f_{k+i} + \sum_{j=1}^k d_{i,j} f_{k+j}, \quad 1 \leq i \leq k.$$

□

Proposition 4.2. Any derivation of the algebra μ_2 has the following matrix form:

$$\mathbb{D} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$\begin{aligned} A &= \sum_{i=1}^{n-2k} (i a_1 + (i-1) b_1) e_{i,i} + \sum_{i=1}^{n-2k-1} \sum_{j=i+1}^{n-2k} a_{j-i+1} e_{i,j}, \\ B &= \sum_{i=1}^{2k} b_i e_{1,i} + \sum_{i=1}^k b_i e_{2,k+i}, \quad C = \sum_{i=1}^k c_i e_{i,n-2k}, \quad D = \begin{pmatrix} D_1 & D_2 \\ 0 & D_3 \end{pmatrix}, \\ D_1 &= \sum_{i=1}^k \sum_{j=2}^k d_{i,j} e_{i,j} + (a_1 + b_1) e_{1,1}, \quad D_3 = D_1 + a_1 \mathbb{E} - \sum_{j=1}^k b_j e_{1,j}, \end{aligned}$$

with $A \in M_{n-2k,n-2k}$, $B \in M_{n-2k,2k}$, $C \in M_{2k,n-2k}$, $D_1, D_2, D_3, \mathbb{E} \in M_{k,k}$ and matrix units $e_{i,j}$.

Proof. The proof is carrying out by straightforward calculation of the derivation property of the algebra μ_2 . □

Proposition 4.3. Any derivation of the algebra μ_3 has the following matrix form:

$$\mathbb{D} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$A = \sum_{i=1}^{n-2k} ((i-1)a_1 + a_2) e_{i,i} + \sum_{i=2}^{n-2k} a_i e_{1,i} + \sum_{i=3}^{n-2k-1} a_i e_{2,i} + \beta e_{2,n-2k} + \sum_{i=3}^{n-2k-1} \sum_{j=i+1}^{n-2k} a_{j-i+2} e_{i,j},$$

$$B = \sum_{i=1}^{2k} b_{1,i} e_{1,i} + \sum_{i=1}^k b_{2,i} e_{2,k+i} + \sum_{i=1}^k b_{1,i} e_{3,k+i}, \quad C = \sum_{i=1}^k c_i e_{i,n-2k}, \quad D = \begin{pmatrix} D_1 & D_2 \\ 0 & (a_1 + a_2)\mathbb{E} + D_1 \end{pmatrix}$$

with $A \in M_{n-2k,n-2k}$, $B \in M_{n-2k,2k}$, $C \in M_{2k,n-2k}$, $D_1, D_2, \mathbb{E} \in M_{k,k}$ and matrix units $e_{i,j}$.

Proof. The proof is carrying out by straightforward calculation of the derivation property of the algebra μ_3 . \square

4.2. Descriptions of algebras $R(\mu_i, 1)$, $i = 1, 2, 3$.

Let us consider the solvable algebra $R(\mu_i, 1) = \mu_i \oplus Q$, $i = 1, 2, 3$, and a basis $\{e_1, e_2, \dots, e_{n-p}, f_1, f_2, \dots, f_p\}$.

We set $\mathcal{F} := \{f_1, f_2, \dots, f_k\}$ and consider the projection of the operator $\mathcal{R}_{x|_{\mathcal{F}}}$ to the space \mathcal{F} (denoted $\varepsilon(\mathcal{R}_{x|_{\mathcal{F}}})$). Let us suppose that there exists a basis of the space \mathcal{F} such that the Jordan form of the operator $\varepsilon(\mathcal{R}_{x|_{\mathcal{F}}})$ can be transformed into a Jordan block J_λ with $\lambda \neq 0$.

Theorem 4.4. *An arbitrary algebra of the family $R(\mu_1, 1)$ admits a basis such that its multiplication table has the following form:*

$$R(\mu_1, 1)(a_2, \dots, a_{n-2k+1}) : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-2k-1, \\ [e_1, f_i] = f_{k+i}, & 1 \leq i \leq k, \\ [e_i, x] = \sum_{j=i+1}^{n-2k} a_{j-i+1} e_j, & 1 \leq i \leq n-2k, \\ [f_i, x] = f_i + f_{i+1}, & 1 \leq i \leq k-1, \\ [f_k, x] = f_k, \\ [f_i, x] = f_i + f_{i+1}, & k+1 \leq i \leq 2k-1, \\ [f_{2k}, x] = f_{2k}, \\ [x, f_i] = -f_i - f_{i+1}, & 1 \leq i \leq k-1, \\ [x, f_k] = -f_k, \\ [x, x] = a_{n-2k+1} e_{n-2k}. \end{cases}$$

Proof. From Proposition 4.1 we have the products in the algebra $R(\mu_1, 1)$:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-2k-1, \\ [e_1, f_i] = f_{k+i}, & 1 \leq i \leq k, \\ [e_1, x] = \sum_{i=1}^{n-2k} a_i e_i + \sum_{i=1}^{2k} b_i f_i, \\ [e_2, x] = 2a_1 e_2 + \sum_{i=3}^{n-2k} a_{i-1} e_i + \sum_{i=1}^k b_i f_{k+i}, \\ [e_i, x] = ia_1 e_i + \sum_{j=i+1}^{n-2k} a_{j-i+1} e_j, & 3 \leq i \leq n-2k, \\ [f_i, x] = c_i e_{n-2k} + \sum_{j=1}^{2k} d_{i,j} f_j, & 1 \leq i \leq k, \\ [f_{k+i}, x] = a_1 f_{k+i} + \sum_{j=1}^k d_{i,j} f_{k+j}, & 1 \leq i \leq k. \end{cases}$$

Let us introduce the notations:

$$[x, e_1] = \sum_{i=1}^{n-2k} \beta_i e_i + \sum_{i=1}^{2k} \beta_{n-2k+i} f_i, \quad [x, f_i] = \sum_{j=1}^{n-2k} \gamma_{i,j} e_j + \sum_{j=1}^{2k} \varphi_{i,j} f_j, \quad 1 \leq i \leq k, \quad [x, x] = \sum_{i=1}^{n-2k} \delta_i e_i + \sum_{i=1}^{2k} \theta_i f_i.$$

Since the space \mathcal{F} forms an abelian algebra, we are in the conditions of Theorem 3.1. Moreover, the products $[e_1, e_1] = e_2$, $[e_1, f_i] = f_{k+i}$ ensure that $e_1, f_i \notin \text{Ann}_r(R(\mu_1, 1))$, $1 \leq i \leq k$. In particular, we

are in the conditions of the algebra R_2 . This implies the existence of a basis of \mathcal{F} such that

$$\begin{aligned} [f_i, x] &= c_i e_{n-2k} + f_i + f_{i+1} + \sum_{j=k+1}^{2k} d_{i,j} f_j, \quad 1 \leq i \leq k-1, \\ [f_k, x] &= c_k e_{n-2k} + f_k + \sum_{j=k+1}^{2k} d_{k,j} f_j, \\ [x, f_i] &= \sum_{j=1}^{n-2k} \gamma_{i,j} e_j - f_i - f_{i+1} + \sum_{j=k+1}^{2k} \varphi_{i,j} f_j, \quad 1 \leq i \leq k-1, \\ [x, f_k] &= \sum_{j=1}^{n-2k} \gamma_{k,j} e_j - f_k + \sum_{j=k+1}^{2k} \varphi_{k,j} f_j, \\ [x, x] &= \sum_{i=1}^{n-2k} \delta_i e_i + \sum_{i=k+1}^{2k} \theta_i f_i. \end{aligned}$$

The equalities

$$\mathcal{L}(x, f_i, e_1) = \mathcal{L}(e_1, x, e_1) = \mathcal{L}(x, e_1, f_i) = 0$$

imply

$$\begin{aligned} \gamma_{i,j} &= \beta_1 = 0, & 1 \leq i \leq k, \quad 1 \leq j \leq n-2k-1, \\ \alpha_1 &= 0, \quad \beta_{n-2k+i} = -b_i, & 1 \leq i \leq k, \\ [x, f_{k+i}] &= 0, & 1 \leq i \leq k. \end{aligned}$$

Since $\{e_2, e_3, \dots, e_{n-2k}, f_{k+1}, \dots, f_{2k}\} \subseteq \text{Ann}_r(R(\mu_1, 1))$ and $[x, x] \in \text{Ann}_r(R(\mu_1, 1))$ while $e_1 \notin \text{Ann}_r(R(\mu_1, 1))$, we conclude $\delta_1 = 0$.

By setting

$$f'_i = f_i - \chi_{i,n-2k} e_{n-2k} - \sum_{j=k+1}^{2k} \psi_{i,j} f_j, \quad 1 \leq i \leq k,$$

where $\chi_{k,n-2k} = \gamma_{k,n-2k}$, $\psi_{k,j} = \varphi_{k,j}$ and parameters $\chi_{i,n-2k}, \psi_{i,j}$ for $1 \leq i \leq k-1$, can be recursively obtained from the products

$$[x, f'_i] + f'_i = -f'_{i+1},$$

we can assume that

$$[x, f_i] = -f_i - f_{i+1}, \quad 1 \leq i \leq k-1, \quad [x, f_k] = -f_k.$$

The equalities $\mathcal{L}(x, f_i, x) = 0$, $1 \leq i \leq k$ we derive $c_i = d_{i,j} = 0$, $1 \leq i \leq k$, $k+1 \leq j \leq 2k$, that is, we obtain

$$[f_i, x] = f_i + f_{i+1}, \quad 1 \leq i \leq k-1, \quad [f_k, x] = f_k.$$

By taking the change of basis element:

$$x' = x - \sum_{i=2}^{n-2k} \beta_i e_{i-1},$$

we obtain $[x', e_1] = -\sum_{i=1}^k b_i f_i + \sum_{i=1}^k \beta_{n-k+i} f_{k+i}$, that is, we can assume that $\beta_i = 0$, $2 \leq i \leq n-2k$.

The equality $\mathcal{L}(x, e_1, x) = 0$ implies

$$\delta_i = 0, \quad 2 \leq i \leq n-2k-1, \quad \beta_{n-k+i} = 0, \quad 1 \leq i \leq k.$$

By putting

$$\begin{aligned} e'_1 &= e_1 + \sum_{i=1}^k \sum_{j=1}^i (-1)^{i+j-1} b_j f_i + \sum_{i=1}^k \sum_{j=1}^i (-1)^{i+j-1} b_{k+j} f_{k+i}, \\ x' &= x + \sum_{i=1}^k \sum_{j=1}^i (-1)^{i+j-1} \theta_{k+j} f_{k+i}, \end{aligned}$$

we deduce

$$b_i = b_{k+i} = \theta_{k+i} = 0, \quad 1 \leq i \leq k.$$

Thus, we obtain the multiplication tables of the algebras $R(\mu_1)(a_2, \dots, a_{n-2k+1})$ of the assertion of the theorem. \square

In the next proposition a necessary and sufficient condition for the existence of an isomorphism between two algebras of the family $R(\mu_1, 1)(a_2, \dots, a_{n-2k+1})$ is established.

Proposition 4.5. *Two algebras $R(\mu_1, 1)'(a'_2, \dots, a'_{n-2k+1})$ and $R(\mu_1, 1)(a_2, \dots, a_{n-2k+1})$ are isomorphic if and only if there exists $A \in \mathbb{C}$ such that*

$$a'_i = \frac{a_i}{A^{i-1}}, \quad 2 \leq i \leq n - 2k + 1.$$

Proof. Let us consider the general change of generator basis elements of the algebra $R(\mu_1, 1)$:

$$\begin{aligned} e'_1 &= \sum_{i=1}^{n-2k} A_i e_i + \sum_{i=1}^{2k} B_i f_i, \\ f'_i &= \sum_{j=1}^{n-2k} C_{i,j} e_j + \sum_{j=1}^{2k} D_{i,j} f_j, \quad 1 \leq i \leq k, \\ x' &= Hx + \sum_{i=1}^{n-2k} E_i e_i + \sum_{i=1}^{2k} F_i f_i. \end{aligned}$$

From the products

$$[e'_i, e'_1] = e'_{i+1}, \quad 1 \leq i \leq n - 2k - 1,$$

of the algebra $R(\mu_1, 1)'$ we derive

$$e'_2 = A_1 \sum_{i=2}^{n-2k} A_{i-1} e_i + A_1 \sum_{i=1}^k B_i f_{k+i}, \quad e'_i = A_1^{i-1} \sum_{j=i}^{n-2k} A_{j-i+1} e_j, \quad 3 \leq i \leq n - 2k.$$

By considering

$$[x', e'_1] = [f'_i, e'_1] = [f'_i, f'_i] = 0, \quad 1 \leq i \leq k,$$

we deduce

$$C_{i,j} = B_i = E_j = 0, \quad 1 \leq i \leq k, \quad 1 \leq j \leq n - 2k - 1.$$

Similarly, from the products $[x', f'_i] = -f'_i - f'_{i+1}$, $1 \leq i \leq k - 1$, and $[x', f'_k] = -f'_k$ we conclude

$$\begin{aligned} H &= 1, \quad D_{i,j} = 0, & 1 \leq j < i \leq k, \\ D_{i,j} &= D_{1,j-i+1}, & 1 \leq i \leq j \leq k, \\ C_{i,n-2k} &= D_{i,k+j} = 0, & 1 \leq i \leq k, \quad 1 \leq j \leq k. \end{aligned}$$

The relations between parameters $\{a'_i\}$ and $\{a_i\}$ follow from the products

$$[e'_1, x'] = \sum_{i=2}^{n-2k} a'_i e'_i, \quad [x', x'] = a'_{n-2k+1} e'_{n-2k}.$$

Namely, we have

$$a'_i = \frac{a_i}{A_1^{i-1}}, \quad 2 \leq i \leq n - 2k + 1.$$

□

Below, we present an analogue of Theorem 4.4 for the family $R(\mu_2, 1)$.

Theorem 4.6. *An arbitrary algebra of the family $R(\mu_2, 1)$ admits a basis such that its multiplication table has the following form:*

$$R(\mu_2, 1)(\alpha, \beta, \gamma) : \begin{cases} \mu_2, \\ [e_1, x] = f_1 + \alpha f_{k+1}, \\ [e_2, x] = e_2 + f_{k+1}, \\ [e_i, x] = (i-1)e_i, & 3 \leq i \leq n-2k, \\ [f_i, x] = -[x, f_i] = f_i + f_{i+1}, & 1 \leq i \leq k-1, \\ [f_k, x] = -[x, f_k] = f_k, \\ [f_{k+1}, x] = f_{k+2}, \\ [f_i, x] = f_i + f_{i+1}, & k+2 \leq i \leq 2k-1, \\ [f_{2k}, x] = f_{2k}, \\ [x, e_1] = -f_1 + \beta \sum_{i=1}^k (-1)^{i-1} f_{k+i}, \\ [x, x] = \gamma f_{k+1}. \end{cases}$$

The classification of the family of algebras $R(\mu_2, 1)(\alpha, \beta, \gamma)$ is presented in the following proposition.

Proposition 4.7. *Any algebra of the family $R(\mu_2, 1)(\alpha, \beta, \gamma)$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$R(\mu_2, 1)(0, 0, 0), \quad R(\mu_2, 1)(0, 0, 1), \quad R(\mu_2, 1)(0, 1, \gamma), \quad R(\mu_2, 1)(1, \beta, \gamma), \quad \beta, \gamma \in \mathbb{C}.$$

Proof. In a similar way as in the proof of Proposition 4.5, we consider the general transformation of generator basis elements:

$$\begin{aligned} e'_1 &= \sum_{i=1}^{n-2k} A_i e_i + \sum_{i=1}^{2k} B_i f_i, \\ f'_i &= \sum_{j=1}^{n-2k} C_{i,j} e_j + \sum_{j=1}^{2k} D_{i,j} f_j, \quad 1 \leq i \leq k, \\ x' &= \sum_{i=1}^{n-2k} E_i e_i + \sum_{i=1}^{2k} F_i f_i + Hx. \end{aligned}$$

Then from the products

$$[e'_i, e'_1] = e'_{i+1}, \quad 1 \leq i \leq n-2k-1, \quad f'_{k+1} = [e'_1, f'_1] - e'_2, \quad f'_{k+i} = [e'_1, f'_i], \quad 2 \leq i \leq k,$$

we derive the rest of new basis elements

$$\begin{aligned} e'_2 &= (A_1 + B_1) \sum_{i=2}^{n-2k} A_{i-1} e_i + A_1 \sum_{i=1}^k B_i f_{k+i}, \\ e'_i &= (A_1 + B_1)^{i-1} \sum_{j=i}^{n-2k} A_{j-i+1} e_j, & 3 \leq i \leq n-2k, \\ f'_{k+1} &= A_1 \sum_{i=1}^k (D_{1,i} - B_i) f_{k+i}, & A_1 \neq 0 \\ f'_{k+i} &= A_1 \sum_{j=2}^k D_{i,j} f_{k+j}, & 2 \leq i \leq k. \end{aligned}$$

By verifying the multiplications of $R(\mu_2, 1)(\alpha', \beta', \gamma')$ in the new basis we obtain the relations between the parameters $\{\alpha', \beta', \gamma'\}$ and $\{\alpha, \beta, \gamma\}$:

$$\alpha' = \frac{\alpha}{A_1}, \quad \beta' = \frac{\beta}{A_1}, \quad \gamma' = \frac{\gamma}{A_1^2}.$$

□

Analogously, we have the description of solvable algebras.

Theorem 4.8. *An arbitrary algebra of the family $R(\mu_3, 1)$ admits a basis such that its multiplication table has the following form:*

$$R(\mu_3, 1)(I) : \left\{ \begin{array}{ll} [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-2k-1, \\ [e_2, f_i] = f_{k+i}, & 1 \leq i \leq k, \\ [e_1, x] = a_1 e_1 + a_{n-2k} e_{n-2k} + b_1 f_1 + b_2 f_{k+1}, \\ [e_2, x] = (a_1 + a_2) e_2 + \sum_{i=4}^{n-2k-1} a_i e_i + \beta e_{n-2k}, \\ [e_3, x] = (2a_1 + a_2) e_3 + \sum_{i=5}^{n-2k} a_{i-1} e_i + b_1 f_{k+1}, \\ [e_i, x] = ((i-1)a_1 + a_2) e_i + \sum_{j=i+2}^{n-2k} a_{j-i+2} e_j, & 4 \leq i \leq n-2k, \\ [f_i, x] = -[x, f_i] = f_i + f_{i+1}, & 1 \leq i \leq k-1, \\ [f_k, x] = -[x, f_k] = f_k, \\ [f_{k+i}, x] = (a_1 + a_2 + 1) f_{k+i} + f_{k+i+1}, & 1 \leq i \leq k-1, \\ [f_{2k}, x] = (a_1 + a_2 + 1) f_{2k}, \\ [x, e_1] = -a_1 e_1 - b_1 f_1 + \sum_{i=1}^k \theta_{n-k+i} f_{k+i}, \\ [x, e_2] = \gamma e_{n-2k}, \\ [x, x] = \delta_1 e_{n-2k-1} + \delta_2 e_{n-2k} + \delta_3 f_{k+1}, \end{array} \right.$$

where $I = \{a_i, b_1, b_2, \beta, \gamma, \delta_1, \delta_2, \delta_3, \theta_{n-k+j}\}$ with $1 \leq i \leq n-2k$, $i \neq 3$, $1 \leq j \leq k$, and for these parameters the following equalities hold true

$$\begin{aligned} (a_1 + a_2)\gamma &= 0, & (n-2k-2)a_1\gamma &= 0, \\ \delta_1 &= -a_1 a_{n-2k}, & a_1 b_2 &= (-1)^{k-1} (a_2 + 1)^k \theta_n, \\ \theta_{n-i} &= (-1)^i (a_2 + 1)^i \theta_n, & 0 \leq i &\leq k-1. \end{aligned} \quad (4.1)$$

We set $J := I \setminus \{a_i, b_1, b_2, \beta, \gamma, \delta_1, \delta_2, \delta_3, \theta_{n-k+j}\}$ with $4 \leq i \leq n-2k$, $1 \leq j \leq k$.

Lemma 4.9. *Let algebras $R(\mu_3, 1)(a_1, a_2, J)$ and $R(\mu_3, 1)(a'_1, a'_2, J')$ are isomorphic. Then*

$$a'_1 = a_1, \quad a'_2 = a_2.$$

Proof. Let us take the general transformation of generators basis elements of the algebra $R(\mu_3)(a_1, a_2, J)$ in the following form:

$$\begin{aligned} e'_1 &= \sum_{i=1}^{n-2k} A_{1,i} e_i + \sum_{i=1}^{2k} B_{1,i} f_i, & e'_2 &= \sum_{i=1}^{n-2k} A_{2,i} e_i + \sum_{i=1}^{2k} B_{2,i} f_i, \\ f'_i &= \sum_{j=1}^{n-2k} C_{i,j} e_j + \sum_{j=1}^{2k} D_{i,j} f_j, \quad 1 \leq i \leq k, & x' &= Hx + \sum_{i=1}^{n-2k} E_i e_i + \sum_{i=1}^{2k} F_i f_i. \end{aligned}$$

Then from the products $[e'_i, e'_1] = e'_{i+1}$, $2 \leq i \leq n-2k-1$, we obtain

$$\begin{aligned} e'_3 &= A_{1,1} \sum_{i=3}^{n-2k} A_{2,i-1} e_i + A_{2,2} \sum_{i=1}^k B_{1,i} f_{k+i}, \\ e'_i &= A_{1,1}^{i-2} \sum_{j=i}^{n-2k} A_{2,j-i+2} e_j, \quad 4 \leq i \leq n-2k. \end{aligned}$$

Moreover, the equalities

$$[e'_1, e'_1] = [e'_2, e'_2] = [f'_i, e'_1] = [e'_3, f'_i] = 0, \quad 1 \leq i \leq k,$$

imply

$$\begin{aligned} A_{1,i} &= 0, & 2 \leq i &\leq n-2k-1, \\ A_{2,1} = 0 &= B_{2,i} = C_{i,j} = 0, & 1 \leq i &\leq k, \quad 1 \leq j \leq n-2k-1. \end{aligned}$$

The rest basis elements f'_{k+i} , $1 \leq i \leq k$, can be obtained from the products $[e'_2, f'_i] = f'_{k+i}$, $1 \leq i \leq k$. Namely, we have

$$f'_{k+i} = A_{2,2} \sum_{j=1}^k D_{i,j} f_{k+j}, \quad 1 \leq i \leq k.$$

Consider $[x', f'_k] = -f'_k$. Then we derive

$$\begin{aligned} (D_{k,i} + D_{k,i+1})H &= D_{k,i+1}, & 1 \leq i \leq k-1, \\ D_{k,1}H &= D_{k,1}, \\ D_{k,i}E_2 &= -D_{k,k+i}, & 1 \leq i \leq k, \\ C_{k,n-2k} &= 0. \end{aligned}$$

If $H \neq 1$, then $D_{k,i} = 0$, $1 \leq i \leq k$, which implies $f'_{2k} = 0$, this is a contradiction. Therefore, $H = 1$ and

$$D_{k,i} = D_{k,k+i} = 0, \quad 1 \leq i \leq k-1.$$

Now from $[x', f'_i] = -f'_i - f'_{i+1}$ with $1 \leq i \leq k-1$, we get restrictions:

$$\begin{aligned} D_{i,j} &= 0, & 1 \leq j < i \leq k, \\ D_{i,j} &= D_{1,j-i+1}, & 1 \leq i \leq j \leq k, \\ C_{i,n-2k} &= 0, & 1 \leq i \leq k-1, \\ D_{i,j}E_2 &= -D_{i,k+j} - D_{i+1,k+j}, & 1 \leq i \leq k-1, \quad 1 \leq j \leq k. \end{aligned}$$

By considering the products $[e'_1, x']$ and $[e'_2, x']$, we deduce

$$a'_1 = a_1, \quad a'_2 = a_2.$$

□

Let us introduce the notations

$$\begin{aligned} I_1 &= \{a_i, \beta, \gamma, \delta_2\}, & 4 \leq i \leq n-2k, \\ I_2 &= \{a_2 = -1, a_i, b_2, \beta, \delta_3, \theta_n\}, & 4 \leq i \leq n-2k, \\ I_3 &= \{a_i, \beta\}, & 2 \leq i \leq n-2k, \quad i \neq 3, a_2 \notin \{-1, 0\}, \\ I_4 &= \{a_1, a_2, a_{n-2k}, b_1, b_2, \delta_1, \delta_2, \delta_3, \theta_{n-k+i}\}, & 1 \leq i \leq k, \end{aligned}$$

where for parameters from the set I_4 we have the relations

$$\begin{aligned} \delta_1 &= -a_1 a_{n-2k}, \\ b_2 &= \frac{(-1)^{k-1} (a_2 + 1)^k \theta_n}{a_1}, \\ \theta_{n-k+i} &= (-1)^{k-i} (a_2 + 1)^{k-i} \theta_n, & 1 \leq i \leq k-1. \end{aligned}$$

Proposition 4.10.

$$R(\mu_3, 1)(I) = \bigcup_{j=1}^4 R(\mu_3, 1)(I_j).$$

Proof. Thanks to Lemma 4.9 for analysis of equalities (4.1) it is sufficient to consider the following cases.

Case 1. Let $a_1 = 0$. Then we get $\delta_1 = \theta_{n-k+i} = 0$, $1 \leq i \leq k-1$ and $a_2 \gamma = (a_2 + 1) \theta_n = 0$.

By applying the change $e'_1 = e_1 + \sum_{i=1}^k (-1)^i b_1 f_i$, we obtain $b_1 = 0$.

Case 1.1. Let $a_2 = 0$. Then $\theta_n = 0$.

By taking the change of basis elements

$$e'_1 = e_1 + \sum_{i=1}^k (-1)^i b_2 f_{k+i}, \quad x' = x + \sum_{i=1}^k (-1)^i \delta_3 f_{k+i},$$

we can assume $b_2 = \delta_3 = 0$. Thus, the subfamily of algebras $R(\mu_3, 1)(I_1)$ is obtained.

Case 1.2. Let $a_2 \neq 0$. Then we have $\gamma = (a_2 + 1) \theta_n = 0$. By setting $x' = x - \frac{\delta_2}{a_2} e_{n-2k}$, we get $\delta_2 = 0$.

Case 1.2.1. Let $a_2 = -1$. Then we obtain the subfamily $R(\mu_3, 1)(I_2)$.

Case 1.2.2. Let $a_2 \neq -1$. Then $\theta_n = 0$.

By putting

$$e'_1 = e_1 + \sum_{i=1}^k (-1)^i \frac{b_2}{(a_2 + 1)^i} f_{k+i}, \quad x' = x + \sum_{i=1}^k (-1)^i \frac{\delta_3}{(a_2 + 1)^i} f_{k+i},$$

we get $b_2 = \delta_3 = 0$.

Thus, we have the class of algebras $R(\mu_3, 1)(I_3)$.

Case 2. Let $a_1 \neq 0$. Then $\gamma = 0$. By applying the change

$$e'_1 = e_1, \quad e'_i = e_i + \sum_{j=i+1}^{n-2k} A_{j-i+2} e_j, \quad 2 \leq i \leq n - 2k,$$

with

$$A_3 = 0, \quad a_3 = 0, \quad A_i = -\frac{1}{(i-2)a_1} \left(\sum_{j=3}^{i-1} A_j a_{i+2-j} + a_i \right), \quad 4 \leq i \leq n - 2k - 1,$$

$$A_{n-2k} = -\frac{1}{(n-2k-2)a_1} \left(\sum_{j=3}^{n-2k-1} A_j a_{n-2k+2-j} + \beta \right),$$

we derive $\beta = a_i = 0$, $4 \leq i \leq n - 2k - 1$.

Therefore, we obtain the class of algebras $R(\mu_3, 1)(I_4)$. \square

Remark 4.11. It should be noted that algebras from different subfamilies $R(\mu_3, 1)(I_j)$, $1 \leq j \leq 4$, are not isomorphic.

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